FUNCTIONAL ABSTRACTIONS OF STOCHASTIC HYBRID SYSTEMS

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Abstract: The verification problem for stochastic hybrid systems is quite difficult. One method to verify these systems is stochastic reachability analysis. Concepts of abstractions for stochastic hybrid systems are needed to ease the stochastic reachability analysis. In this paper, we set up different ways to define abstractions for stochastic hybrid systems, which preserve the parameters of stochastic reachability. A new concept of stochastic bisimulation is introduced and its connection with equivalence of stochastic processes is established.

Keywords: Markov models, probability function, verification, hybrid systems

1. INTRODUCTION

The investigation of stochastic hybrid systems (SHS) has recently received significant attention (Bujorianu, 2004), (Blom, 2003), (Pola *et al.*, 2003). The need for probabilistic modelling is motivated mainly by the partial knowledge about very complex systems, as well as, by the possibility of simplifying deterministic models introducing probabilistic reasoning. There are several dimensions of probabilistic reasoning over hybrid systems: (1) probabilistic quantification of discrete transitions; (2) stochastic reasoning about continuous evolution; (3) probabilistic aspects of the interaction between continuous and discrete dynamics. Because of these multiple dimensions, there is need to develop approaches towards the verification of SHS. Until now, concrete steps to solve the verification problems were made only for ad-hoc or particular classes of SHS models.

The novelty of the approach presented in this paper is to study the verification problem of SHS as bisimulation for a stochastic realization problem and to define abstractions of SHS that 'preserve' the probabilities used to define the stochastic reachability problem. Our abstractions focus on achieving numerical evaluation of the probability bounds even with the price of sacrificing the model expressivity. Ideally, abstractions for stochastic reachability analysis would provide state spaces in the real line, making available efficient algorithms from numerical analysis.

An important problem in the development of SHS is the preservation of stochastic properties. An abstraction can affect the probabilities and implicitly, the reachability analysis. Therefore, it is crucial to develop a theory of 'bisimulation' of SHS that preserves the relevant reach set probabilities of SHS considered. The aim of this paper is to initiate the development of such bisimulations/abstractions for SHS in a general setting.

The paper is structured as follows. In the next section, we give the motivation of this work and the formulation of the problem. Then, we present a short background on SHS, their semantics as stochastic processes and different operator methods which can be used to characterize stochastic processes. The main body of the paper is constituted by Sect.4. The main goal of this section is to define concepts of stochastic bisimulation and abstractions for SHS, which do not conduct to the equivalence of the SHS realizations. The attempt to define bisimulation that preserves different stochastic parametrizations of the SHS realizations is a wrong track, since it is nothing else, but another way, to present equivalence of stochastic processes. This section is divided in five parts. In the first subsection we define a space reduction technique based on quadratic forms associated to stochastic processes. The use of this technique is motivated by the formulation of the stochastic reachability for SHS. The goal of stochastic reachability analysis is to measure the set of trajectories, which reach a given target set until a given time horizon. Often, in practice, the target set is described as a level set for a 'nice' real-valued function defined on the whole state space. This function might be a norm, an observably function, or a weight function, etc. Applying this function to the paths of stochastic process that constitutes the SHS realization gives rise to a new stochastic process with the state space in the real line. Even the given process is Markovian and the above function has some measurability properties the new process might not be Markovian (Rogers and Pitman, 1981). To 'hide' this drawback we use the induced quadratic form, which intuitively is the composition between the quadratic form of the initial process and the above function. Usually, the new quadratic form has good properties such that there exists a Markov process associated to it. This new process is the best candidate to represent an abstraction of the initial process. We call it *functional abstraction*. In this way, the SHS realizations are 'approximated' by stochastic processes with a much smaller state space. The stochastic parameters of the induced process can be easily derived. Using this method, we formally define, in the following subsections, new concepts of stochastic bisimulation and functional abstractions of SHS. We have to underlie that the functional abstractions 'preserve' the continuous and jumping parts of an SHS. The paper ends with some conclusions.

2. PROBLEM FORMULATION

In this section we briefly present the stochastic reachability problem for stochastic hybrid systems and starting from it we derive the main ideas for defining abstractions for SHS.

Stochastic Reachability. Consider $M = (x_t, P_x)$ being a strong Markov process, the realization of a stochastic hybrid system (see definitions below). For this strong Markov process we address a verification problem consisting of the following stochastic reachability problem. Given a set $A \in \mathcal{B}(X)$ and a time horizon $T > 0$, let us to define (Bujorianu and Lygeros, 2003), (Bujorianu, 2004):

$$Reach_T(A) = \{\omega \in \Omega \mid \exists t \in [0,T] : x_t(\omega) \in A\}$$
$$Reach_\infty(A) = \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in A\}. \quad (1)$$

These two sets are the sets of trajectories of M, which reach the set A (the flow that enters A) in the interval of time $[0,T]$ or $[0,\infty)$. The reachability problem consists of determining the probabilities of such sets. The reachability problem is well-defined, i.e. $Reach_T(A)$, $Reach_\infty(A)$ are indeed measurable sets. Then the probabilities of reach events are: $P(T_A < T)$ or $P(T_A < \infty)$, where $T_A = \inf\{t > 0 | x_t \in A\}$ and P is a probability on the measurable space $(\Omega, \mathcal{F})$ of the elementary events associated to M. P can be chosen to be P_x (if we want to consider the trajectories, which start in x) or P_μ (if we want to consider the trajectories, which start in x_0 given by the distribution μ). Recall that $P_\mu(A) = \int P_x(A)d\mu$, $A \in \mathcal{F}$.

Usually a target set A is a level set for a given function $F : X \to \mathbb{R}$, i.e. $A = \{x \in X | F(x) > l\}$. The probability of the set of trajectories, which hit A until time horizon $T > 0$ can be expressed as $P[\sup\{F(x_t) > l \mid t \in [0,T]\}]$.

Problem Formulation. Define a new stochastic process $M^\#$ such that the reach set probabilities are preserved.

Idealy, the above argument shows that $F(x_t)$ would represent the best candidate for defining a possible abstraction for M, which preserves the reach set probabilities. The main difficulty is that $F(x_t)$ is a Markov process only for special choices of F (Rogers and Pitman, 1981). The problem is how to choose F well.

3. PRELIMINARIES

In this section we give the necessary background for stochastic hybrid systems, their semantics, some stochastic analysis tools and Dirichlet forms.

3.1 Stochastic Hybrid Systems

Many practical systems such as automobiles, chemical processes, and autonomous vehicles are best described by dynamics that comprise continuous state evolution within a mode of operation and discrete transitions from one mode to another, either controlled or autonomous. Such systems often interact with their environment in the presence of uncertainty and variability. SHS can model complex dynamics, uncertainty, multiple modes of operations and support high-level control specifications that are required for design of (semi-)autonomous applications. Several modelling paradigms for SHS have been already proposed in literature. A stochastic hybrid scheme that allows the continuous flows at each discrete location to be characterized by stochastic differential equations is described in (Hu *et al.*, 2000). An extension of this model that satisfies the strong Markov property is presented in (Blom, 2003). Methods to study the reachability problem for SHS have been addressed in (Bujorianu and Lygeros, 2003). Dynamically Coloured Petri Nets and Communicating Piecewise Deterministic Markov Processes as compositional specifications for SHS in (Everdij and Blom, 2005) with emphasis on modelling concurrency. Applications of SHS to large distributed systems have been studied for air traffic management systems (Pola *et al.*, 2003) and for communication networks (Hespanha, 2004).

3.2 Semantics of Stochastic Hybrid Systems

The executions of a stochastic hybrid system H form a stochastic process. Let us consider $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$, the realization (or semantics) of H. Under mild assumptions on the parameters of H, M can be viewed as a family of Markov processes with the state space $(X, \mathcal{B})$, where X is the union of modes and $\mathcal{B}$ is its Borel σ-algebra. Let $\mathcal{B}^b(X)$ be the lattice of bounded positive measurable functions on X. The meaning of the elements of M can be found in any source treating continuous-parameter Markov processes (Davis, 1993). Suppose we have given a σ-finite measure μ on $(X, \mathcal{B})$.

In the following, some operator characterizations (used, in this paper, to define abstractions for SHS) of stochastic processes are given.

Operator Semigroup / Resolvent. Let $p_t(x, A) = P_x(x_t \in A)$, $A \in \mathcal{B}$ be the transition probability function. The meaning of this is the probability that, if $x_0 = x$, x_t will lie in the set A. The operator semigroup $\mathcal{P}$ is defined by $P_t f(x) = \int f(y) p_t(x, dy) = E_x f(x_t), \forall x \in X$, where E_x is the expectation w.r.t. P_x. The operator semigroup $(P_t)_{t>0}$ is, in fact, the collection of all first order moments, which can be associated with the family of random variables $\{x_t | t > 0\}$. The operator resolvent $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ associated with $\mathcal{P}$ is $V_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt$, $x \in X$. Let denote by V the initial operator V_0 of $\mathcal{V}$, which is known as the *kernel operator* of the Markov process M. The operator resolvent $(V_\alpha)_{\alpha \geq 0}$ is the Laplace transform of the semigroup. The strong *generator* $\mathcal{L}$ is the derivative of P_t at $t = 0$. Let $D(\mathcal{L}) \subset \mathcal{B}_b(X)$ be the set of functions f for which the following limit exists (denoted by $\mathcal{L}f$): $\lim_{t \searrow 0} \frac{1}{t}(P_t f - f)$. A *quadratic form* $\mathcal{E}$ can be associated to the generator of a Markov process in a natural way. Let $L^2(X, \mu)$ be the space of square integrable μ-measurable extended real valued functions on X, w.r.t. the natural inner product $< f, g >_\mu = \int f(x) g(x) d\mu(x)$. A quadratic form $\mathcal{E}$ is defined as a closed form: $\mathcal{E}(f, g) = - < \mathcal{L}f, g >_\mu, f \in D(\mathcal{L}), g \in L^2(X, \mu)$. This leads to another way of parameterizing Markov processes. Instead of writing down a generator one starts with a quadratic form. As in the case of a generator it is typically not easy to fully characterize the domain of the quadratic form. For this reason one starts by defining a quadratic form on a smaller space and showing that it can be extended to a closed form in subset of $L^2(\mu)$. When the Markov process can be initialized to be stationary, the measure μ is typically this stationary distribution (see (Davis, 1993), p.111). More generally, μ does not have to be a finite measure. If M is a right Markov process then $\mathcal{E}$ is a regular Dirichlet form (Ma and Rockner, 1990), (Fukushima, 1980).

Dirichlet Forms. A *coercive closed form* (Albeverio *et al.*, 1993) is a quadratic form $(\mathcal{E}, D(\mathcal{E}))$ with $D(\mathcal{E})$ dense in $L^2(X, \mu)$, which satisfies the: (i) closeness axiom, i.e. its symmetric part is positive definite and closed in $L^2(X, \mu)$, (ii) continuity axiom (*Sector condition*). $\mathcal{E}$ is called *Dirichlet form* if, in addition, it satisfies the third axiom: (iii) contraction condition (*Dirichlet property*), i.e. $\forall u \in D(\mathcal{E})$, $u^* = u^+ \wedge 1 \in D(\mathcal{E})$ and $\mathcal{E}(u \pm u^*, u \mp u^*) \geq 0$. See (Ma and Rockner, 1990), (Fukushima, 1980), (Albeverio *et al.*, 1993).

Let $(\mathcal{L}, D(\mathcal{L}))$ be the generator of a coercive form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(X, \mu)$, i.e. the unique closed linear operator on $L^2(X, \mu)$ such that $1 - \mathcal{L}$ is onto, $D(\mathcal{L}) \subset D(\mathcal{E})$ and $\mathcal{E}(u, v) = < -\mathcal{L}u, v >$ for all $u \in D(\mathcal{L})$ and $v \in D(\mathcal{E})$. Let $(T_t)_{t>0}$ be the strongly continuous contraction semigroup on $L^2(X, \mu)$ generated by $\mathcal{L}$ and $(G_\alpha)_{\alpha>0}$ the corresponding strongly continuous contraction semigroup. A right process M with the state space X is *associated* with a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(X, \mu)$ if the semigroup (P_t) of the process M is a μ-version of the form semigroup (T_t). Only those Dirichlet forms (called *quasi-regular Dirichlet forms*), which satisfy some regularity conditions can be associated with some right Markov processes and viceversa (Th.1.9 (Albeverio *et al.*, 1993)). Prop. 4.2 from (Albeverio *et al.*, 1993) states that two right Markov processes M and M' with state space X associated with a common quasi-regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ are stochastically equivalent (Ma and Rockner, 1990), (Fukushima, 1980). That means a quasi-regular Dirichlet form characterizes a class of stochastically equivalent right Markov processes.

4. ABSTRACTIONS OF STOCHASTIC HYBRID SYSTEMS

The idea is to apply a "state space reduction" technique based on the general 'induced Dirichlet forms' method to achieve abstractions for SHS. With this technique, the realizations of SHS are 'approximated' by a one-dimensional stochastic process with a much smaller state space.

4.1 Induced Dirichlet Forms

First, we define the concept of *induced Dirichlet form* (introduced in (Iscoe and McDonald, 1990) only for symmetric Dirichlet forms) and prove some properties (relations between generators, operator semigroups, kernel operators) of this concept, which will be used further in defining a new concept of stochastic bisimulation between Markov processes.

Let $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ be a right Markov process with the state space X. Now assume that X is a Lusin space (i.e. it is homeomorphic to a Borel subset of a compact metric space) and $\mathcal{B}(X)$ or $\mathcal{B}$ is its Borel σ-algebra. Note that for the majority of the stochastic hybrid system models the state space is a Lusin space (Pola *et al.*, 2003). Assume also that μ is a σ-finite measure on $(X, \mathcal{B})$ and μ is a stationary measure of the process M. Let $X^\#$ another Lusin space (with $\mathcal{B}^\#$ its Borel σ-algebra) and $F : X \to X^\#$ be a measurable function. Let $\sigma(F)$ be the sub-σ-algebra of $\mathcal{B}$ generated by F. If μ is a probability measure then the projection operator between $L^2(X, \mathcal{B}, \mu)$ and $L^2(X, \sigma(F), \mu)$ is the conditional expectation $E_\mu[\cdot|F]$. Recall that E_μ is the expectation defined w.r.t. P_μ. We denote by $\mu^\#$ the image of μ under F, i.e. $\mu^\#(A^\#) = \mu(F^{-1}(A^\#))$, for all $A^\# \in \mathcal{B}^\#$. In general, anything associated with $X^\#$ will carry an #-superscript in this section.

Let $\mathcal{E}$ be the Dirichlet form on $L^2(X, \mu)$ associated to M. F induces a form $\mathcal{E}^\#$ on $L^2(X^\#, \mu^\#)$ by

$$\mathcal{E}^\#(u^\#, v^\#) = \mathcal{E}(u^\# \circ F, v^\# \circ F);$$

for $u^\#, v^\# \in D[\mathcal{E}^\#]$, where

$$D[\mathcal{E}^\#] = \{u^\# \in L^2(X^\#, \mu^\#) | u^\# \circ F \in D[\mathcal{E}]\}.$$

It can be shown (Prop.1.4 (Iscoe and McDonald, 1990)), under a mild condition on the conditional expectation operator $E_\mu[\cdot|F]$ that $\mathcal{E}^\#$ is a Dirichlet form. If, in addition, $\mathcal{E}^\#$ is quasi-regular then we can associate with it a right Markov process $M^\# = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t^\#, P_x^\#)$ be a right Markov process with the state space $X^\#$. The process $M^\#$ is called the *induced Markov process* w.r.t. to the proper map F. If the image of M under F is a right Markov process then $x_t^\# = F(x_t)$. The process $M^\#$ might have some different interpretations like a refinement of discrete transitions structure, or an approximation of continuous dynamics or an abstraction of the entire process. It is difficult to find a practical condition to impose on F, which would guarantee that $\mathcal{E}^\#$ is also quasi-regular. To circumvent this problem, it is possible to restrict the original domain $D[\mathcal{E}^\#]$ and impose some regularity conditions on F (Iscoe and McDonald, 1990).

Assumption 1. Suppose that $\mathcal{E}^\#$ is a quasi-regular Dirichlet form.

Let $(\mathcal{L}, D(\mathcal{L}))$ and $(\mathcal{L}^\#, D(\mathcal{L}^\#))$ be the generators of $\mathcal{E}$ and $\mathcal{E}^\#$, respectively.

Proposition 1. Under assumption 1, the generators $\mathcal{L}$ and $\mathcal{L}^\#$ are related as follows

$$\mathcal{L}(u^\# \circ F) = \mathcal{L}^\# u^\# \circ F, \forall u^\# \in D(\mathcal{L}^\#).$$

Theorem 2. Under assumption 1, for all $A^\# \in \mathcal{B}^\#(X^\#)$ and for all $t > 0$ we have $p_t^\#(Fx, A^\#) = p_t(x, F^{-1}(A^\#))$, where $(p_t^\#)$ and (p_t) are the transition functions of $M^\#$ and M, respectively.

Corollary 3. Under assumption 1, the semigroups $(P_t^\#)$ and (P_t) of $M^\#$ and M are related by $P_t^\# u^\# \circ F = P_t(u^\# \circ F), \forall u^\# \in \mathcal{B}^b(X^\#)$.

Remark 1. In the terminology of (Bujorianu *et al.*, 2005), we can say that $M^\#$ simulates M.

4.2 Stochastic Bisimulation

In this subsection we define a new concept of stochastic bisimulation for SHA. This concept is defined as measurable relation (Strubbe and Schaft, 2005), which induces equivalent Dirichlet forms on the quotient spaces. In comparison with (Strubbe and Schaft, 2005), in defining stochastic bisimulation, we do not impose the equivalence of the quotient processes, which might not have Markovian properties (Rogers and Pitman, 1981), but we impose the equivalence of the induced Markov processes (that can differ from the quotient processes) associated with the induced Dirichlet forms.

Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be Lusin spaces and let $\mathcal{R} \subset X \times Y$ be a relation such that $\Pi^1(\mathcal{R}) = X$ and $\Pi^2(\mathcal{R}) = Y$. We define the equivalence relation on X that is induced by the relation $\mathcal{R} \subset X \times Y$, as the transitive closure of $\{(x, x') | \exists y$ s.t. $(x, y) \in \mathcal{R}$ and $(x', y) \in \mathcal{R}\}$. Analogously, the induced (by $\mathcal{R}$) equivalence relation on Y can be defined. We write $X/_{\mathcal{R}}$ and $Y/_{\mathcal{R}}$ for the sets of equivalence classes of X and Y induced by $\mathcal{R}$. We denote the equivalence class of $x \in X$ by $[x]$. Let $\mathcal{B}^{\#}(X) = \mathcal{B}(X) \cap \{A \subset X |$ if $x \in A$ and $[x] = [x']$ then $x' \in A\}$ be the collection of all Borel sets, in which any equivalence class of X is either totally contained or totally not contained. It can be checked that $\mathcal{B}^{\#}(X)$ is a σ-algebra. Let $\pi_X : X \to X/_{\mathcal{R}}$ be the mapping that maps each $x \in X$ to its equivalence class and let $\mathcal{B}(X/_{\mathcal{R}}) = \{A \subset X/_{\mathcal{R}} | \pi_X^{-1}(A) \in \mathcal{B}^{\#}(X)\}$. Then $(X/_{\mathcal{R}}, \mathcal{B}(X/_{\mathcal{R}}))$, which is a measurable space, is called the quotient space of X w.r.t. $\mathcal{R}$. The quotient space of Y w.r.t. $\mathcal{R}$ is defined in a similar way. We define a bijective mapping $\psi : X/_{\mathcal{R}} \to Y/_{\mathcal{R}}$ as $\psi([x]) = [y]$ if $(x, y) \in \mathcal{R}$ for some $x \in [x]$ and some $y \in [y]$. We say that the relation $\mathcal{R}$ is *measurable* if X and Y if for all $A \in \mathcal{B}(X/_{\mathcal{R}})$ we have $\psi(A) \in \mathcal{B}(Y/_{\mathcal{R}})$ and vice versa, i.e. ψ is a homeomorphism (Strubbe and Schaft, 2005). Then the real measurable functions defined on $X/_{\mathcal{R}}$ can be identified with those defined on $Y/_{\mathcal{R}}$ through the homeomorphism ψ. We can write $\mathcal{B}^b(X/_{\mathcal{R}}) \overset{\psi}{\cong} \mathcal{B}^b(Y/_{\mathcal{R}})$. These functions can be thought of as real functions defined on X or Y measurable w.r.t. $\mathcal{B}^{\#}(X)$ or $\mathcal{B}^{\#}(Y)$.

Assumption 2. Suppose that $X/_{\mathcal{R}}$ and $Y/_{\mathcal{R}}$ with the topologies induced by projection mappings are Lusin spaces.

Let M and W be two right Markov processes with the state spaces X and Y. Assume that μ (resp. ν) is a stationary measure of the process M (resp. W). Let $\mu/_{\mathcal{R}}$ (resp. $\nu/_{\mathcal{R}}$) the image of μ (resp. ν) under π_X (resp. π_Y). Let $\mathcal{E}$ (resp. $\mathcal{F}$) the quasi-regular Dirichlet form corresponding to M (resp. W). The equivalence of the induced processes can be used to define a new bisimulation between Markov processes, as follows.

Under assumptions 1 and 2, a measurable relation $\mathcal{R} \subset X \times Y$ is a *bisimulation between M and W* if the mappings π_X and π_Y define the same induced Dirichlet form on $L^2(X/_{\mathcal{R}}, \mu/_{\mathcal{R}})$ and $L^2(Y/_{\mathcal{R}}, \nu/_{\mathcal{R}})$, respectively. This bisimulation definition states that M and W are bisimilar if $\mathcal{E}/_{\mathcal{R}} = \mathcal{F}/_{\mathcal{R}}$. Here, $\mathcal{E}/_{\mathcal{R}}$ (resp. $\mathcal{F}/_{\mathcal{R}}$) is the induced Dirichlet form of $\mathcal{E}$ (resp. $\mathcal{F}$) under the mapping π_X (resp. π_Y). Clearly, this can be possible iff $\mu/_{\mathcal{R}} = \nu/_{\mathcal{R}}$.

Assumption 3. Suppose that $\mathcal{E}/_{\mathcal{R}}$ and $\mathcal{F}/_{\mathcal{R}}$ are quasi-regular Dirichlet form.

Denote the Markov process associated to $\mathcal{E}/_{\mathcal{R}}$ (resp. $\mathcal{F}/_{\mathcal{R}}$) by $M/_{\mathcal{R}}$ (resp. $W/_{\mathcal{R}}$).

Proposition 4. Under assumptions 1,2, 3, M and W are stochastic bisimilar w.r.t. $\mathcal{R}$ iff the processes $M/_{\mathcal{R}}$ and $W/_{\mathcal{R}}$ are $\mu/_{\mathcal{R}}$-equivalent.

Let H and H' be two stochastic hybrid systems, with the realizations M and W, strong Markov processes defined on the state spaces $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$, respectively. H and H' are *bisimilar* if there exists a bisimulation relation under which their realizations M and W are bisimilar.

4.3 Weak Stochastic Bisimulation

The way to define the concept of stochastic bisimulation, in the previous subsection, presents two main difficulties: assumptions 2 and 3. It seems difficult to find a practical condition to impose on $\mathcal{R}$, which would guarantee that these two assumptions are fulfilled. There exist conditions that ensure that the quotient space of an analytic space is again analytic, but it might be difficult to find out necessary conditions on $\mathcal{R}$, which ensure that the quotient space of a Lusin space is again Lusin. When we pass from the quasi-regular Dirichlet forms of the initial processes it might be hard to deal with conditions on the projection mappings π_X and π_Y, which assure the regularity of the induced Dirichlet forms.

With these arguments in mind, in this subsection we will introduce a weaker version of stochastic bisimulation and the concept of *functional abstraction* of a stochastic hybrid system.

Let M and W be two right Markov processes with the state spaces X and Y, as in the previous subsections. Suppose we have given two weight (measurable) functions $F : X \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ (F or G can be the function used to define the target sets in context of the stochastic reachability problem. Let $\mathcal{E}^{\#}$ (resp. $\mathcal{F}^{\#}$) be the induced Dirichlet form of $\mathcal{E}$ (resp. $\mathcal{F}$) through the mapping F (resp. G).

Assumption 4. Suppose that $\mathcal{E}^{\#}$ and $\mathcal{F}^{\#}$ are quasi-regular Dirichlet forms.

M and W are *(weak) stochastic bisimilar* if the induced Dirichlet forms are equal, i.e. $\mathcal{E}^{\#} = \mathcal{F}^{\#}$.

The advantage of this new definition is that we do not need anymore the property to be Lusin of the quotient spaces. On the other hand the induced processes are one-dimensional stochastic processes whose state spaces are much smaller ones.

The stochastic reachability definition gives the idea to introduce the following concept of functional abstraction for SHS.

Given a right Markov process M defined on the Lusin state space $(X, \mathcal{B})$, and $F : X \to \mathbb{R}$ a measurable weight function, suppose that Ass.1 is fulfilled. The process $M^{\#}$ associated to the induced Dirichlet form $\mathcal{E}^{\#}$ under function F is called a *functional abstraction of* M.

Let H a stochastic hybrid system and M its realization. Suppose that M is a right Markov process defined on the Lusin state space $(X, \mathcal{B})$.

Any stochastic hybrid system $H^{\#}$ whose realization is a functional abstraction of M is called a *functional abstraction of* H.

Let M be a right Markov process, thought as the realization of an SHA, H.

Proposition 5. If M is a diffusion (resp. jump process) then any functional abstraction $M^{\#}$ of M is a diffusion (resp. jump process).

Since the realization of an SHA is a stochastic process, which can be viewed an interleaving between some diffusion processes and a jump process (Bujorianu and Lygeros, 2004) we can write the following corollary of Prop.5.

Proposition 6. Any functional abstraction of an SHA is again an SHA.

5. CONCLUSIONS

In this paper, motivated by problem of stochastic reachability for SHS we have introduced

• a new concept of stochastic bisimulation for SHS from a functional viewpoint, i.e. this bisimulation is defined using the parameters, which appear in the reachability problem formulation and preserves the bounds of reach set probabilities;

• functional abstractions for SHS, defined again from a functional perspective.

The main tool used in defining of these new concepts is constituted by the quadratic forms associated to the realizations of the SHS. The quadratic form technique makes possible to obtain abstractions of SHS realizations, which are one-dimensional stochastic processes with a much smaller state space. The meaning of the induced stochastic process might be different depending on the context: refinement of discrete transitions structure, or approximation of continuous dynamics or, finally, abstraction of the entire process.

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